TMA4170 Fourier Analysis

Fourier inversion

$$\mathcal{F}[f](\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx$$

$$\mathcal{F}^{-1}[g](x) = \int_{-\infty}^{\infty} g(\xi)\, e^{+2\pi i x \xi}\, d\xi \qquad \left(= \mathcal{F}[g](-x) = \mathcal{F}[g(-\xi)](x) \right)$$

$\mathcal{F}$ is invertible on $S(\mathbb{R})$ with inverse $\mathcal{F}^{-1}$:

Theorem: $f, g \in S(\mathbb{R})$

(a) $\mathcal{F}[f], \mathcal{F}^{-1}[g] \in S(\mathbb{R})$

(b) $f(x) = \mathcal{F}^{-1}[\mathcal{F}[f]](x), \quad g(\xi) = \mathcal{F}[\mathcal{F}^{-1}[g]](\xi)$

Convolutions in $S(\mathbb{R})$

$$f, g \in S(\mathbb{R}) \implies f * g \in S(\mathbb{R}) \text{ and } \widehat{f * g} = \hat{f} \cdot \hat{g}$$

Plancherel identity in $S(\mathbb{R})$

$$f, g \in S(\mathbb{R}) \implies \begin{cases} \|f\|_2 = \|\hat{f}\|_2 \\ (f, g)_2 = (\hat{f}, \hat{g})_2 \end{cases}$$

$\|f\|_2^2 = \int_{\mathbb{R}} |f|^2$

$(f, g)_2 = \int_{\mathbb{R}} f \cdot \bar{g}$

$\mathcal{F}$ is an isometry w.r.t. $\|\cdot\|_2$ in $S(\mathbb{R})$.

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Fourier transform in $C_m(\mathbb{R})$ and $L^1(\mathbb{R})$

Properties:

(i) $\hat{f} \in C_0(\mathbb{R})$, $|\hat{f}(\xi)| \underset{|\xi|\to\infty}{\longrightarrow} 0$, $\|\hat{f}\|_\infty \le \|f\|_1$

(ii) $f, f' \in C_m(L^1) \implies \widehat{f'}(\xi) = 2\pi i \xi \cdot \hat{f}(\xi)$

(iii) $f, g \in C_m(L^1) \implies \widehat{f * g} = \hat{f} \cdot \hat{g}$

(iv) $\mathcal{F}^{-1}[g(y)](x) = \mathcal{F}[g](-x) = \mathcal{F}[g(-y)](x)$

(v) $K(x) = e^{-\pi x^2} \implies \hat{K}(\xi) = K(\xi) = e^{-\pi \xi^2}$

Fourier inversion in $C_m(\mathbb{R})$ and $L^1(\mathbb{R})$

① (Global) inversion when $\hat{f} \in L^1$:

$$f, \hat{f} \in C_m(L^1) \implies f(x) = \mathcal{F}^{-1}[\hat{f}](x)$$

② "Fejer" type result when $\hat{f} \notin L^1$:

$$f \in C_m(L^1) \implies f(x) = \lim_{\delta \to 0} \mathcal{F}^{-1}[\hat{K}_\delta \cdot \hat{f}](x), \qquad \hat{K}_\delta(\xi) = e^{-\pi\delta\xi^2}$$

③ Pointwise inversion where f smooth:

$f \in PC_m(L^1)$ and $f(x^\pm)$, $D^\pm f(x)$ exists

$$\implies \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{2}\left(f(x^-) + f(x^+)\right)$$